

Answers

Part I

Q1. The plane should follow the parabola 飛機須沿拋物線運動。

$$x = v_0 t \cos \theta, y = v_0 t \sin \theta - \frac{1}{2} g t^2 \quad (3 \text{ points})$$

Q2 (6 points)

$$I \ddot{\omega} = T$$

The center of the rod will not move in the horizontal direction 杆中心在水平方向不動。

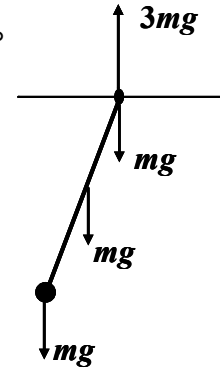
$$I = \frac{ml^2}{12} + 2m\left(\frac{l}{2}\right)^2 = \frac{7}{12} ml^2 \quad (2 \text{ points})$$

There are two ways to find the torque. 找力距的方法有兩種。

Method-1 方法-1

The forces acting upon the rod are shown. The torque to the center of the rod is 由如圖力的分析，可得

$$T = -(2mg + mg) \frac{l}{2} \theta = -mg \frac{3l}{2} \theta \quad (2 \text{ points})$$



Method-2 方法-2

Given a small angle deviation θ from equilibrium, the potential energy is 給定一個角度的小位移 θ ，勢能為

$$U = mg \frac{3l}{2} (1 - \cos \theta) \simeq mg \frac{3l}{4} \theta^2.$$

$$T = -\frac{\partial U}{\partial \theta} = -mg \frac{3l}{2} \theta \quad (2 \text{ points})$$

$$\text{Finally, 最後得 } \frac{7ml^2}{12} \ddot{\theta} = \frac{3mgl}{2} \theta \Rightarrow \omega = \sqrt{\frac{18g}{7l}}. \quad (2 \text{ points})$$

Q3 (6 points)

(a) The bound current density on the disk edge is 盤邊的束縛電流密度為

$$K = -M \times n = -M, \quad (1 \text{ point})$$

The bound current is 束縛電流 $\Rightarrow I = Jd = -Md$, (1 point)

The B-field is 磁場為 $B(z=h) = \frac{\mu_0 R^2 I}{2(R^2 + h^2)^{\frac{3}{2}}} = -\frac{\mu_0 R^2 Md}{2(R^2 + h^2)^{\frac{3}{2}}}$ (1 point)

(b) The bound current density is $K = -M \times n = -M$, which is on the side wall of the cylinder. (1 point)

柱側面上的束縛電流密度為 $K = -M \times n = -M$

The problem is then the same as a long solenoid. Take a small Ampere loop we get $B = \mu_0 K = \mu_0 M$ inside;

為求一長線圈的磁場，取一小閉合路徑，得介質內 $B = \mu_0 K = \mu_0 M$ (1 point)

Outside 介質外 $B = 0$ (1 point)

Q4 (5 points)

Each unit charge in the slab experiences the Lorentz force $-vB y_0$. (1 point)

The problem is then the same as a dielectric slab placed between two parallel conductor plates that carry surface charge density $\pm\sigma$, and $\frac{\sigma}{\epsilon_0} = vB$. In such

case, the electric displacement is $D = \sigma$. $P = D - \epsilon_0 E = D - \frac{D}{\epsilon} = \epsilon_0 vB (\frac{\epsilon - 1}{\epsilon})$. (2 point)

介質內單位電荷受力 $-vB y_0$ 。問題變成兩電荷面密度為 $\pm\sigma$, and $\frac{\sigma}{\epsilon_0} = vB$ 的

導電板間充滿介質。因此 $D = \sigma$. $P = D - \epsilon_0 E = D - \frac{D}{\epsilon} = \epsilon_0 vB (\frac{\epsilon - 1}{\epsilon})$.

Finally, the bound surface charge is $\sigma_b = P = \epsilon_0 vB (\frac{\epsilon - 1}{\epsilon})$. The upper surface carries positive bound charge, and the lower surface carries negative charge. (1 point)

最後得束縛電荷密度 $\sigma_b = P = \epsilon_0 vB (\frac{\epsilon - 1}{\epsilon})$ ，上表面帶正電，下表面帶負電。

The electric field is $\vec{E} = \frac{\sigma_b}{\epsilon_0} \vec{y}_0 = vB (\frac{\epsilon - 1}{\epsilon}) \vec{y}_0$, which is along the y-direction

(opposite to the Lorentz force). (1 point)

電場為 $\vec{E} = \frac{\sigma_b}{\epsilon_0} \vec{y}_0 = vB \left(\frac{\epsilon - 1}{\epsilon} \right) \vec{y}_0$, 與 Lorentz 力方向相反。

Q5. (10 points)

- (a) Because of the spherical symmetry, the E-field and the current density J are all along the radial direction. In steady condition, the electric current I through any spherical interfaces must be equal. Since the area of the sphere is proportional to r^2 , J must be proportional to $1/r^2$. So let $\vec{J} = \frac{K}{r^2} \hat{r}$, where K is a constant to be determined, and the expression holds in both media. (1 point)

由對稱性可知，電場和電流密度 J 須沿半徑方向。穩態時，流過每個包住球心的球面的電流相等，因此 J 與 $1/r^2$ 成正比。設在兩介質裏 $\vec{J} = \frac{K}{r^2} \hat{r}$ ， K 為待定常數。

In medium-1 介質-1, $E_1 = \frac{1}{\sigma_1} J = \frac{1}{\sigma_1} \frac{K}{r^2}$, and the voltage drop from R_1 to R_2 is

$$V_1 = \frac{1}{\sigma_1} K \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad (1 \text{ point})$$

介質-1, $E_1 = \frac{1}{\sigma_1} J = \frac{1}{\sigma_1} \frac{K}{r^2}$ ，從 R_1 到 R_2 的電壓為 $V_1 = \frac{1}{\sigma_1} K \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$

Likewise, in medium-2, $E_2 = \frac{1}{\sigma_2} \frac{K}{r^2}$, and the voltage drop from R_2 to R_3 is

$$V_2 = \frac{1}{\sigma_2} K \left(\frac{1}{R_2} - \frac{1}{R_3} \right). \quad (1 \text{ point})$$

同樣，在介質-2, $E_2 = \frac{1}{\sigma_2} \frac{K}{r^2}$ ，從 R_2 到 R_3 的電壓為 $V_2 = \frac{1}{\sigma_2} K \left(\frac{1}{R_2} - \frac{1}{R_3} \right)$

The total voltage drop between R_1 and R_3 is 總電壓 $V = V_1 + V_2$.

$$\text{So 得 } \frac{1}{K} = \frac{1}{V} \left[\frac{1}{\sigma_1} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{1}{\sigma_2} \left(\frac{1}{R_2} - \frac{1}{R_3} \right) \right]$$

The current is 電流為 $I = 4\pi R_1^2 J(R_1) = 4\pi K$. (1 point)

The electric displacement in media are $D_{1,2} = \frac{\epsilon_{1,2}}{\sigma_{1,2}} J$, so the charge on the inner,

outer, and boundary shells are $4\pi K \frac{\epsilon_1}{\sigma_1}$, $-4\pi K \frac{\epsilon_2}{\sigma_2}$, and $4\pi K \left(\frac{\epsilon_2}{\sigma_2} - \frac{\epsilon_1}{\sigma_1} \right)$,

respectively.

介質-1 裏電位移 $D_{1,2} = \frac{\epsilon_{1,2}}{\sigma_{1,2}} J$, 因此在各面上的電荷為 $4\pi K \frac{\epsilon_1}{\sigma_1}$, $-4\pi K \frac{\epsilon_2}{\sigma_2}$,

$$4\pi K \left(\frac{\epsilon_2}{\sigma_2} - \frac{\epsilon_1}{\sigma_1} \right)。$$

(b) Due to symmetry, the electric field is of the form $\vec{E} = \frac{K}{r^2} \hat{r}$, so

$$V = K \left(\frac{1}{R_1} - \frac{1}{R_3} \right), \text{ and } K = V \cdot \frac{R_3 - R_1}{R_1 R_3}. \text{ (2 point)}$$

由對稱性可知，電場為 $\vec{E} = \frac{K}{r^2} \hat{r}$ ，因此電壓為 $V = K \left(\frac{1}{R_1} - \frac{1}{R_3} \right)$ ，得

$$K = V \cdot \frac{R_3 - R_1}{R_1 R_3}。$$

The current densities in the two hemispheres are $J_1 = \sigma_1 E$ and $J_2 = \sigma_2 E$. The

total current is $I = 2\pi R_1^2 \cdot J_1(R_1) + 2\pi R_1^2 \cdot J_2(R_1) = 2\pi K(\sigma_1 + \sigma_2)$. (2 point)

上下半部的電流密度為 $J_1 = \sigma_1 E$ ， $J_2 = \sigma_2 E$ 。總電流為

$$I = 2\pi R_1^2 \cdot J_1(R_1) + 2\pi R_1^2 \cdot J_2(R_1) = 2\pi K(\sigma_1 + \sigma_2)。$$

The total free charge is $Q_1 = 2\pi \epsilon_0 \epsilon_1 R_1^2 \cdot E_1(R_1) = 2\pi K \epsilon_0 \epsilon_1$ on the upper half and

$Q_2 = 2\pi K \epsilon_0 \epsilon_2$ on the lower half of the inner shell. On the outer shell the

charges are negative of the corresponding ones of the inner shell. (1 point)

內球面上半部總自由電荷為 $Q_1 = 2\pi \epsilon_0 \epsilon_1 R_1^2 \cdot E_1(R_1) = 2\pi K \epsilon_0 \epsilon_1$ ，下半部總自

由電荷為 $Q_2 = 2\pi K \epsilon_0 \epsilon_2$ 。外球面的電荷與內球面相反。

Q6. (12 points)

a) $P_h - P_{h+dh} = \rho g dh \Rightarrow \frac{dP}{dh} = -\rho g$ (1 point)

$$PV^{\frac{7}{5}} = \text{Constant} \Rightarrow P = C\rho^{\frac{7}{5}} \Rightarrow \rho = \left(\frac{P}{C}\right)^{\frac{5}{7}}, \text{ where } (C = \frac{P_0}{\rho_0^{\frac{7}{5}}}) \quad (1 \text{ point})$$

Combine these two equations, 合併兩式得

$$\frac{dP}{dh} = -\left(\frac{P}{C}\right)^{\frac{5}{7}} g \Rightarrow P = P_0 \left(1 - \frac{2\rho_0 g h}{7P_0}\right)^{\frac{7}{2}} \quad (2 \text{ point})$$

$$\rho = \rho_0 \left(1 - \frac{2\rho_0 g h}{7P_0}\right)^{\frac{5}{2}}, \text{ and } T = T_0 \left(1 - \frac{2\rho_0 g h}{7P_0}\right). \quad (2 \text{ point})$$

b) $TV^{\frac{2}{5}} = \text{Constant}, \frac{T}{\rho^{\frac{2}{5}}} = \text{Constant}$ (1 point)

The density at 40°C is $\frac{\rho^{\frac{2}{5}}}{313} = \frac{1.18^{\frac{2}{5}}}{293} \Rightarrow \rho = 1.18 \times \left(\frac{313}{293}\right)^{\frac{5}{2}}$ (1 point)

40°C 的空氣密度為 $\frac{\rho^{\frac{2}{5}}}{313} = \frac{1.18^{\frac{2}{5}}}{293} \Rightarrow \rho = 1.18 \times \left(\frac{313}{293}\right)^{\frac{5}{2}}$

The fraction of water vapor at 40°C at sea level 40°C 的水蒸汽分壓為,

$$\eta_1 = \frac{55.35}{760} \times 90\% \quad (1 \text{ point})$$

The fraction of water at 5°C at high altitude 5°C 的水蒸汽分壓為,

$$\eta_2 = \frac{6.5}{760 \left(\frac{278}{313}\right)^{\frac{7}{2}}} \quad (1 \text{ point})$$

Rain 下雨量

$$= (\eta_1 - \eta_2) \times \rho \times V = \left(\frac{55.35}{760} \times 90\% - \frac{6.5}{760 \left(\frac{278}{313}\right)^{\frac{7}{2}}}\right) \times 1.18 \times \left(\frac{313}{293}\right)^{\frac{5}{2}} = 0.07 \text{ kg} \quad (1 \text{ point})$$

高度 : $278 = 313 \left(1 - \frac{2 \times 1.18 \times 9.8 \times h}{7 \times 1.03 \times 10^5}\right) \Rightarrow h = 3500 \text{ m} \quad (1 \text{ point})$

Q7. (8 points)

i) $P \times E$ (1 point)

ii) $P \times E$ (1 point)

iii) $D = \varepsilon_0 E + P \Rightarrow P = (\varepsilon - 1)\varepsilon_0 E \Rightarrow P \times E = 0$ (1 point)

iv) $P = D - \varepsilon_0 E = \varepsilon_0[(\varepsilon_X - 1)E_X X_0 + (\varepsilon_Y - 1)E_Y Y_0]$ (1 point)

$$\bar{T} = \frac{1}{2} \text{Re}(\overrightarrow{P \times E^*}) = \frac{1}{2} \text{Re}(\overrightarrow{D \times E^*}) = \frac{1}{2} \varepsilon_0 E_0^2 (\varepsilon_X - \varepsilon_Y) \cos(\Delta k z), \text{ (1 point)}$$

Where 其中 $\Delta k \equiv (\sqrt{\varepsilon_X} - \sqrt{\varepsilon_Y}) \frac{\omega}{c}$. (1 point)

v) $\bar{T} = \frac{1}{2} \varepsilon_0 (\varepsilon_X - \varepsilon_Y) E_0^2 \int_0^d \cos(\Delta k z) dz = \frac{1}{2\Delta k} \varepsilon_0 (\varepsilon_X - \varepsilon_Y) E_0^2 \sin(\Delta k d)$. (1 point)

Maximum occurs when 最大值在 $\Delta k d = \frac{\pi}{2} \Rightarrow d = \frac{\pi}{2\Delta k}$. (1 point)

Part II

Q1. (6 points)

a) $k_x a = n\pi \Rightarrow k_x = \frac{n\pi}{a}, n = 1, 2, 3, \dots$ (1 point)

b) $k_y 2b\pi = 2m\pi \Rightarrow k_y = \frac{m}{b}, m = 1, 2, 3, \dots$ (3 points)

c) $W = \frac{1239}{2\pi} \sqrt{k_x^2 + k_y^2} \Leftrightarrow \frac{1239}{2\pi} \sqrt{\frac{n^2 \pi^2}{a^2} + \frac{m^2}{b^2}} = 10^{12}$
 $\Rightarrow \frac{1239}{2\pi} \frac{m}{b} < 10^{12} \Rightarrow b > \frac{1239}{\pi} \times 10^{-12} \approx 2 \times 10^{-10} \text{ nm}$ (2 points)

Q2. (22 points)

i) $I = m_2 l^2 + \frac{m_1}{3} l^2$ (2 points)

$$I \dot{\omega} = F(t) \cdot l \Leftrightarrow (m_2 l^2 + \frac{m_1}{3} l^2) \ddot{\theta} = -\theta k l^2 \Rightarrow \omega_0 = \sqrt{\frac{k}{m_2 + \frac{m_1}{3}}} \quad (2 \text{ points})$$

ii) $(m_2 l + \frac{m_1 l}{3}) \ddot{x} = F_1 \cos(\omega_1 t) l - x k l$ (2 points)

$$x = A_1 \cos(\omega_1 t + \phi_1)$$

$$\Rightarrow -(\frac{m_1}{3} + m_2) A_1 \omega_1^2 \cos(\omega_1 t + \phi_1) = F_1 \cos(\omega_1 t) - A_1 \cos(\omega_1 t + \phi_1) k$$

$$\Rightarrow \phi_1 = 0, A_1 = \frac{F_1}{k - (\frac{m_1}{3} + m_2) \omega_1^2} \quad (2 \text{ points})$$

iii) $(m_2 + \frac{m_1}{3}) \ddot{x} = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t) - x k$ (2 points)

$$x = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t$$

$$\Rightarrow A_1 = \frac{F_1}{k - (\frac{m_1}{3} + m_2) \omega_1^2}, A_2 = \frac{F_2}{k - (\frac{m_1}{3} + m_2) \omega_2^2} \quad (2 \text{ points})$$

iv) $x = \sum_n A_n \cos \omega_n t \Rightarrow A_n = \frac{F_n}{k - (\frac{m_1}{3} + m_2) \omega_n^2}$ (2 points)

v) $V_{out} = V_0 e^{i\omega t} = \frac{R + iL\omega}{\frac{1}{i\omega C} + R + iL\omega} V_0 e^{i\omega t} = \frac{iR\omega C - L\omega^2 C}{1 - L\omega^2 C + iR\omega C} V_0 e^{i\omega t}$

$$|V_{out}|^2 = V_0^2 \frac{(R\omega C)^2 + (L\omega^2 C)^2}{(1 - L\omega^2 C)^2 + (R\omega C)^2} \quad (1 \text{ point})$$

To make the denominator minimum, we should have 使分母最小

$$1 - LC\omega^2 = 0 \Rightarrow L = \frac{1}{C\omega^2} \quad (1 \text{ point})$$

(vi) For a given L , only the signal with $\omega_n = \sqrt{\frac{1}{CL}}$ in the answer of (iv) can pass

through the filter, (1 point) and the output is proportional to A_n . (1 point) By varying L one selects different ω_n , and the output is proportional to the selected A_n . (1 point)

From (iv), the maximum A_n is the one when $k = (\frac{m_1}{3} + m_2)\omega_n^2$. (1 point) So as L is varied, one finds a particular $L_{\max}$ at which the signal peaks, and $k = (\frac{m_1}{3} + m_2)CL_{\max}$. (1 point)

給定 L , 則只有頻率為 $\omega_n = \sqrt{\frac{1}{CL}}$ 的信號可通過濾波器, (1 point) 其大小正比於 A_n .

(1 point) L 的變化等於選擇不同的 ω_n 的信號. (1 point) 由(iv)知, $k = (\frac{m_1}{3} + m_2)\omega_n^2$ 時 A_n 最大. (1 point) 因此可得使輸出信號最大的電感 $L_{\max}$, 並有 $k = (\frac{m_1}{3} + m_2)CL_{\max}$.

Correct sketch is a flat line with a peak at $L_{\max}$ 正確的簡圖是一平線, 在 $L_{\max}$ 有一尖峰 (1 point)

Q3. (22 points)

i) Only the charge $q = \frac{r^3}{R^3}Ze$ that is inside the sphere r will have a non-zero net force on the nucleus. (2 points)

只有 $q = \frac{r^3}{R^3}Ze$ 這麼多的負電荷對原子核的合力不為零。

$$\frac{1}{4\pi\epsilon_0} \frac{Zeq}{r^2} = ZeE_0 \Rightarrow \frac{1}{4\pi\epsilon_0} \frac{rZe}{R^3} = E_0 \Rightarrow r = \frac{4\pi\epsilon_0 E_0 R^3}{Ze} \quad (2 \text{ points})$$

$$\text{ii) } Zm_e \ddot{r} = ZeE(t) - \frac{Ze}{4\pi\epsilon_0} \times \frac{rZe}{R^3} \quad (2 \text{ points})$$

$$E(t) = A \cos(\omega t), r = B \cos(\omega t + \varphi) \quad (1 \text{ point})$$

$$\Rightarrow -Zm_e B \omega^2 \cos(\omega t + \varphi) = ZeA \cos(\omega t) - \frac{Z^2 e^2}{4\pi\epsilon_0} \times \frac{B \cos(\omega t + \varphi)}{R^3}$$

$$\Rightarrow \varphi = 0, B = \frac{eA}{\frac{Ze^2}{4\pi\epsilon_0} \cdot \frac{1}{R^3} - m_e \omega^2} \quad (2 \text{ points})$$

$$\Rightarrow p = Zer = \frac{Ze^2 A}{\frac{Ze^2}{4\pi\epsilon_0} \cdot \frac{1}{R^3} - m_e \omega^2} \cos(\omega t) \quad (1 \text{ point})$$

$$\text{iii) } P(t) = Np = \frac{NZe^2}{\frac{Ze^2}{4\pi\epsilon_0} \cdot \frac{1}{R^3} - m_e \omega^2} E(t) \quad (2 \text{ points})$$

$$\text{iv) } P = CE_{ext} = (\epsilon - 1)\epsilon_0 E_{total} = (\epsilon - 1)\epsilon_0 (E_{ext} + E_{self}) \quad (2 \text{ points})$$

The electric field E in a uniform polarized ball with polarization vector P can be calculated by considering the surface bound charge $P \cos \theta$ distributed on the ball.

均匀極化球內的電場 E 可由球面的束縛電荷 $P \cos \theta$ 求得。

$$E_{self} = \frac{1}{4\pi\epsilon_0} \iint \frac{P \cos \theta (-\cos \theta)}{R^2} R^2 \sin \theta d\theta d\varphi = -\frac{P}{3\epsilon_0} \quad (2 \text{ points})$$

$$E_{self} = -\frac{P}{3\epsilon_0} = \frac{1}{3} CE_{ext} \quad (1 \text{ point})$$

$$\Rightarrow CE_{ext} = (\epsilon - 1)\epsilon_0 (E_{ext} - \frac{CE_{ext}}{3\epsilon_0})$$

$$\Rightarrow 3\epsilon_0 \frac{\epsilon - 1}{\epsilon + 2} = C = \frac{NZe^2}{\frac{Ze^2}{4\pi\epsilon_0} \cdot \frac{1}{R^3} - m_e \omega^2}$$

$$\Rightarrow K(\omega) = C = \frac{NZe^2}{\frac{Ze^2}{4\pi\epsilon_0} \cdot \frac{1}{R^3} - m_e \omega^2} = \frac{N(Ze)^2}{Zm_e(\omega_0^2 - \omega^2)},$$

where $\omega_0^2 \equiv \frac{4\pi\epsilon_0 R^3}{Zm_e(Ze)^2}$ is the resonant frequency of an atom (2 points)

and $n = 2$

$\omega_0^2 \equiv \frac{4\pi\epsilon_0 R^3}{Zm_e(Ze)^2}$ 是原子的共振頻率。

v) For air K is very small and $\epsilon(\omega)$ is close to 1.

So the L-L relation becomes

$$\epsilon(\omega) - 1 = \frac{1}{\epsilon_0} K(\omega) \quad (1 \text{ point})$$

Air density decreases with height. So refractive index decreases with height.
(1 point)

Light rays are bent in such condition. (1 point)

空氣的 $\epsilon(\omega)$ 接近 1, K 很小, 上述結果簡化為 $\epsilon(\omega) - 1 = \frac{1}{\epsilon_0} K(\omega)$ 。空氣的密度

隨高度減小, 因此折射率也減小, 形成如圖所示的光線彎曲。

